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A Note on the Pure Theory of Consumer's Behaviour

By P. A. Samuelson

From its very beginning the theory of consumer's choice has marched steadily towards greater generality, sloughing off at successive stages unnecessarily restrictive conditions. From the time of Gossen to our own day we have seen the removal of (*a*) the assumption of linearity of marginal utility; (*b*) the assumption of independence of utilities; (*c*) the assumption of the measurability of utility in a cardinal sense; and (*d*) even the assumption of an integrable field of preference elements.

The discrediting of *utility* as a psychological concept robbed it of its only possible virtue as an *explanation* of human behaviour in other than a circular sense, revealing its emptiness as even a construction. As a result the most modern theory confines itself to an analysis of indifference elements, budgetary equilibrium being defined by equivalence of price ratios to respective indifference slopes.

Consistently applied, however, the modern criticism turns back on itself and cuts deeply. For just as we do not claim to know by introspection the behaviour of utility, many will argue we cannot know the behaviour of ratios of marginal utilities or of indifference directions.

Why should one believe in the *increasing rate of marginal substitution*, except in so far as it leads to the type of demand functions in the market which seem plausible? Even on the advanced front we are confronted with this dilemma —either the argument with respect to indifference varieties is circular or to many people inadmissible (at least without further demonstration).

Hence, despite the fact that the notion of utility has been repudiated or ignored by modern theory, it is clear that much of even the most modern analysis shows vestigial traces of the utility concept. Thus, to any person not acquainted with the history of the subject, the exposition of the theory of consumer's behaviour in the formulation of Hicks and Allen[1] would seem indirect. The introduction and meaning of the marginal rate of substitution as an entity independent of any psychological, introspective implications would be, to say the least, ambiguous, and would seem an artificial convention in the explanation of price behaviour. (This would be particularly so in the many-commodity, non-integrable case.)

I propose, therefore, that we start anew in direct attack upon the problem, dropping off the last vestiges of the utility analysis. This does not preclude the introduction of utility by any who may care to do so, nor will it contradict the results attained by use of related constructs. It is merely that the analysis can be carried on more directly, and from a different set of postulates.

All that follows shall relate to an idealised individual—not necessarily, however, the rational *homo-economicus*. I assume in the beginning as known, i.e., empirically determinable under ideal conditions, the amounts of n economic goods which will be purchased per unit time by an individual faced with the prices of these goods and with a given total expenditure. It is assumed that prices are taken as given parameters not subject to influence by the individual.

Postulate I. Mathematically we assume as known the following single-valued functions:

$$\left.\begin{aligned} \psi_1 &= h^1(p_1, \ldots p_n, I) \\ &\vdots \\ \psi_n &= h^n(p_1, \ldots p_n, I) \end{aligned}\right\} \qquad (1.0)$$

subject to

$$\psi_1 p_1 + \psi_2 p_2 + \ldots \psi_n p_n = \sum_{i=1}^{n} \psi_i p_i = I \qquad (1.1)$$

Since we have $(n+1)$ equations and only n dependent variables, it is obvious that we may suppress one of the equations as redundant. We may rewrite our first assumption omitting the first equation in (1.0).

$$\left.\begin{aligned} &\psi_i = h^i(p_1, \ldots p_n, I) \qquad (i=2, \ldots n) \\ &\sum_{i=1}^{n} \psi_i p_i - I = 0 \end{aligned}\right\} \qquad (1.2)$$

[1] Hicks and Allen, "A Reconsideration of the Theory of Value," ECONOMICA, February and May, 1934.

Thus, confronted with a given set of prices and with a given income, our idealised individual will always choose the same set of goods. For mathematical convenience we assume that all our functions and their derivatives of the desired order are continuous with no singularities in the region under discussion.

Postulate II. We further assume that the consumer's behaviour is independent of the units in which prices are expressed. More specifically, if we multiply all prices and income by the same positive quantity, the amounts taken will remain the same.[1]

Mathematically our functions in set (1.2) are all homogeneous of order zero; or

$$\left.\begin{aligned} &\psi_i = h^i(\lambda p_1, \ldots \lambda p_n, \lambda I) \qquad (i=2, \ldots n) \\ &\sum_{i=1}^{n} \psi_i \lambda p_i - \lambda I = 0 \end{aligned}\right\} \quad (1.21)$$

Defining $\lambda = \frac{1}{p_1}$, these may be rewritten

$$\left.\begin{aligned} \psi_i &= h^i\left(1, \frac{p_2}{p_1}, \ldots \frac{p_n}{p_1}, \frac{I}{p_1}\right) \qquad (i=2, \ldots n) \\ &= g^i\left(\frac{p_2}{p_1}, \ldots \frac{p_n}{p_1}, \frac{I}{p_1}\right) \qquad (i=2, \ldots n) \\ \psi_1 &+ \sum_{i=2}^{n} \frac{p_i}{p_1} \psi_i - \frac{I}{p_1} = 0 \end{aligned}\right\} \quad (1.3)$$

This is equivalent to using p_1 as our *numéraire*, i.e., setting its price equal to unity.

Let us define $\beta_i = \frac{p_i}{p_1}$, $(i=2, \ldots n)$ and $I^0 = \frac{I.}{p_1}$

We may rewrite (1.3) in the form

$$\left.\begin{aligned} &\psi_i = g^i(\beta_2, \ldots \beta_n, I^0) \qquad (i=2, \ldots n) \\ &\psi_1 + \sum_{i=2}^{n} \beta_i \psi_i - I^0 = 0 \end{aligned}\right\} \quad (2.0)$$

Thus far we have not assumed that anything is known concerning the form or structural properties of our demand functions. Merely knowing that there will be a unique reaction to a given price and income situation puts no restrictions on that reaction. Fortunately, as I shall later

[1] This homogeneity assumption has been challenged by Mr. Keynes with respect to a different problem. For the pure theory of consumer's behaviour it is probably without objection. In any case it is always implicitly made.

show, it is possible to develop suitable restrictions so that our theory is more than formal.

Before doing so, it is convenient to develop certain additional relations. Substituting for I^0_i its equivalent, we may rewrite set (2.0) in the following implicit form:

$$g^i(\beta_2, \ldots \beta_n, \psi_1 + \sum_{i=2}^{n} \beta_i \psi_i) - \psi_i = 0 \quad (i = 2, \ldots n) \quad (3.0)$$

These are $(n-1)$ implicit functions involving $(2n-1)$ variables. Assuming that the conditions of the implicit function theorem are met in the region under discussion (and from later assumptions this will be likely), we may solve explicitly for the $(n-1)$ β's in terms of the n quantities; i.e.,

$$\beta_i = \beta_i(\psi_1, \ldots \psi_n) \qquad (i = 2, \ldots n) \qquad (4.0)$$

As before, nothing is known of the form of these functions in the absence of additional assumptions. It remains only to introduce some appropriate restriction to limit the form of our various functions.

To do so, I suggest the consideration of matters lying close to the modern theory of index numbers.

Let us consider an initial price and income situation.

$$(p_1, \ldots p_n, I)$$

Corresponding to this set there is a given set of consumer's goods bought.

$$(\psi_1, \ldots \psi_n)$$

Now consider a second set of prices and income.

$$(p_1', \ldots p_n', I')$$

and

$$(\psi_1', \ldots \psi_n')$$

I introduce a bracket notation to indicate the following sum:

$$[\psi p] = \psi_1 p_1 + \psi_2 p_2 + \ldots \psi_n p_n = \sum_{i=1}^{n} \psi_i p_i$$

or

$$[\psi p'] = \psi_1 p_1' + \psi_2 p_2' + \ldots \psi_n p_n' = \sum_{i=1}^{n} \psi_i p_i'$$

Suppose now that we combine the prices of the first position with the batch of goods bought in the second. The total cost of such a batch would be

$$[\psi' p] = \sum_{i=1}^{n} \psi_i' p_i \qquad (5.0)$$

If this cost is less than or equal to the actual expenditure in the first period when the first batch of goods was actually bought, then it means that the individual could have purchased the second batch of goods with the price and income of the first situation, but did not choose to do so. That is, the first batch (x) was selected over (x'). We may express this symbolically by saying

$$[\psi' p] \leqq [\psi p] \tag{5.11}$$

implies

$$(\psi') \textcircled{<} (\psi) \tag{5.12}$$

The last symbol is merely an expression for the fact that the first batch was selected over the second. A reversal of the inequality sign could have the meaning that the second batch was selected over the first.

By analogous reasoning

$$[\psi p'] \leqq [\psi' p'] \tag{5.21}$$

implies

$$(\psi) \textcircled{<} (\psi') \tag{5.22}$$

By the usual rules of logic

$$(\psi) \not\textcircled{<} (\psi') \tag{5.31}$$

implies

$$[\psi p'] > [\psi' p'] \tag{5.32}$$

since the negation of a consequence negates the antecedent.

Postulate III. I assume the following consistency in our idealised individual's behaviour. In any two price and income situations and corresponding quantities of consumer's goods given by equations (1.0) the individual must always behave consistently in the sense that (5.12) and (5.22) cannot hold simultaneously. Symbolically, we may write Postulate III

$$(\psi') < (\psi) \tag{6.01}$$

implies

$$(\psi) \nless (\psi') \tag{6.02}$$

In words this means that if an individual selects batch one over batch two, he does not at the same time select two over one. The meaning of this is perfectly clear and will probably gain ready acquiescence. In any case the denial of this restriction would render invalid all of the former analysis of consumer's behaviour and the theory of index numbers as shown later.

Surprisingly enough, from this assumption and from this alone, it is possible to develop important restrictions on our demand functions.

Let us suppose

$$[\psi' p] = [\psi p] \tag{7.01}$$

This implies from (5.11) and (5.12)

$$(\psi') \; \textcircled{<} \; (\psi) \tag{7.02}$$

which implies from (6.01) and (6.02)

$$(\psi) \; \textcircled{$\nless$} \; (\psi') \tag{7.03}$$

which implies from (5.31) and (5.32)

$$[\psi p'] > [\psi' p'] \tag{7.04}$$

To summarise the argument

$$[\psi' p] = [\psi p] \tag{7.01}$$

implies

$$[\psi p'] > [\psi' p']^1 \tag{7.04}$$

Without loss of generality let us write

$$\psi_i' = \psi_i + \Delta\psi_i ; \quad p_i' = p_i + \Delta p_i$$

(7.01) and (7.04) become

$$[(\psi + \Delta\psi)p] = [\psi p] \tag{8.01}$$

and

$$[\psi(p + \Delta p)] > [(\psi + \Delta\psi)(p + \Delta p)] \tag{8.02}$$

Dropping the bracket notation

$$\sum_{i=1}^{n} (\psi_i + \Delta\psi_i)p_i = \sum_{i=1}^{n} \psi_i p_i \tag{8.01}$$

$$\sum_{i=1}^{n} \psi_i(p_i\Delta p_i) > \sum_{i=1}^{n} (\psi_i + \Delta\psi_i)(p_i + \Delta p_i) \tag{8.02}$$

Setting the price of the first good equal to unity and using the notation previously defined

$$\psi_1 + \Delta\psi_1 + \sum_{i=2}^{n} (\psi_i + \Delta\psi_i)\beta_i = \psi_1 + \sum_{i=2}^{n} \psi_i\beta_i \tag{9.01}$$

$$\psi_1 + \sum_{i=2}^{n} \psi_i(\beta_i + \Delta\beta_i) > \psi_1 + \Delta\psi_1 + \sum_{i=2}^{n} (\psi_i + \Delta\psi_i)(\beta_i + \Delta\beta_i) \tag{9.02}$$

[1] This means in the theory of index numbers that if the Laspeyres quantity index is equal to one, then the Paasche quantity index must be less than one. This provides a possible statistical check on the hypotheses underlying various index number studies.

By algebraic cancellation of terms this becomes

$$\Delta\psi_1 + \sum_{i=2}^{n} \beta_i \Delta\psi_i = 0 \qquad (10.01)$$

$$\Delta\psi_1 + \sum_{i=2}^{n} \beta_i \Delta\psi_i + \sum_{i=2}^{n} \psi_i \Delta\beta_i < 0 \qquad (10.02)$$

By substitution from the first of these expressions we may simplify the second.

$$\sum_{i=2}^{n} \Delta\psi_i \Delta\beta_i < 0 \qquad (10.03)$$

It will be noted that these are identically the conditions derived by Georgescu-Roegen in his admirable article,[1] conditions here derived exclusively from our three simple postulates. In our notation, however, the β_i stand for actual price ratios and not hypothetical indifference directions.

From now on we may parallel Georgescu-Roegen's mathematical analysis. Letting our second point approach the first we have the following condition holding in the limit:

$$d\psi_1 + \sum_{i=2}^{n} \beta_i d\psi_i = 0 \qquad (11.01)$$

$$\sum_{i=2}^{n} \sum_{j=1}^{n} \beta_{i,j} d\psi_i d\psi_j < 0 \qquad (11.02)$$

where $\beta_{i,j} = \frac{\partial \Sigma_i}{\partial \psi_j}$ in the functions defined in (4.0). These two conditions together imply, as Georgescu-Roegen has shown, that the following quadratic form be negative definite.

$$\sum_{i=2}^{n} \sum_{j=1}^{n} (\beta_{i,j} - \beta_j \beta_{i,1}) \xi_i \xi_j < 0 \qquad (12.0)$$

This requires that the principal minors beginning with the third of the following determinant be alternately negative and positive.

$$- \begin{vmatrix} 0 & 1 & \beta_2 & \cdots & \beta_n \\ 1 & 0 & \beta_{2,1} & \cdots & \beta_{n,1} \\ \beta_2 & \beta_{2,1} & 2\beta_{2,2} & \cdots & \beta_{2,n} + \beta_{n,2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \beta_n & \beta_{n,1} & \beta_{n,2} + \beta_{2,n} & \cdots & 2\beta_{n,n} \end{vmatrix} \qquad (12.01)$$

[1] "The Pure Theory of Consumer's Behaviour," *The Quarterly Journal of Economics*, Vol. L, August, 1936, pp. 545–593.

i.e.,

$$-\begin{vmatrix} 0 & 1 & \beta_2 \\ 1 & 0 & \beta_{2,1} \\ \beta_2 & \beta_{2,1} & 2\beta_{2,2} \end{vmatrix} < 0 ; -\begin{vmatrix} 0 & 1 & \beta_2 & \beta_3 \\ 1 & 0 & \beta_{2,1} & \beta_{3,1} \\ \beta_2 & \beta_{2,1} & 2\beta_{2,2} & \beta_{2,3}+\beta_{3,2} \\ \beta_3 & \beta_{3,1} & \beta_{3,2}+\beta_{2,3} & 2\beta_{3,3} \end{vmatrix} < 0 \quad (12.02)$$

In the absence of the assumption of integrability the quadratic form in (12.0) is not symmetrical, which apparently Allen overlooked in his exposition.

Concerning the question of integrability I have little to say. I cannot see that it is really an important problem, particularly if we are willing to dispense with the utility concept and its vestigial remnants. Thus, Georgescu-Roegen's demonstration of the spiral-like behaviour of indifference varieties projected upon the budget plane at the point of equilibrium, while acute and illuminating, in no way changes matters if the point of view advanced here is accepted. The only possible interest that integrability can have (except to those who have an historical attachment to the utility concept) would be in providing us with additional knowledge concerning a certain reciprocal relation, namely,

$$\beta_{i,j} - \beta_j\beta_{i,1} = \beta_{j,i} - \beta_i\beta_{j,1} \qquad (i, j = 2, \ldots n) \qquad (13.0)$$

But it is this very implication which makes it doubtful and subject to refutation under ideal observational conditions, although I have little faith in any attempts to verify this statistically. I should strongly deny, however, that for a rational and consistent individual integrability is implied, except possibly as a matter of circular definition.

It remains now only to translate our restrictive conditions on the functions (4.0) into direct restrictions on our demand functions.[1] We may rewrite (8.01) and (8.02) in the form

$$\sum_{i=1}^{n} p_i d\psi_i = 0 \qquad (14.0)$$

$$\sum_{i=1}^{n} dp_i d\psi_i < 0 \qquad (14.1)$$

where not all dx_i vanish, i.e., where not all prices are allowed to vary in the same proportion.

From (1.0)

$$d\psi_i = \sum_{j=1}^{n} , h_j^i dp_j + h_I^i dI \qquad (i = 1, \ldots n) \qquad (14.2)$$

[1] I should like to express my indebtedness to Mr. Rollin Bennett for his valuable aid in much that follows.

But

$$dI = \sum_{j=1}^{n} \psi_j dp_j + \sum_{j=1}^{n} p_j d\psi_j \tag{14.3}$$

However from (14.0), this may be written

$$dI = \sum_{j=1}^{n} \psi_j dp_j \tag{14.4}$$

Hence,

$$d\psi_i = \sum_{j=1}^{n} (h_j^i + h_I^i \psi_i) dp_j \qquad (i = 1, \ldots n) \tag{14.5}$$

Therefore (14.1) becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (h_j^i + h_I^i \psi_j) dp_i dp_j < 0 \tag{15.0}$$

for not all prices varying proportionately.

Defining[1]

$$\alpha_{ij} = h_j^i + h_I^i \psi_j + h_i^j + h_I^j \psi_j = \alpha_{ji} \tag{15.1}$$

our direct conditions on the demand function are that the n^2 quadratic form be negative *semi-definite*[2]

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} dp_i dp_j \leqq 0 \tag{16.0}$$

where the equality holds only for all prices changing in proportion. Because of our homogeneity condition, the n^2 determinant corresponding to this form vanishes identically,

$$|\alpha_{ij}| \equiv 0 \tag{17.0}$$

Our final conditions are that every $(n-1)^2$ minor of this determinant must have been formed from the coefficients of an $(n-1)^2$ negative definite form. Briefly, it is necessary that the following minors alternate in sign for any ordering of variables.

$$\alpha_{ii} < 0; \quad \begin{vmatrix} \alpha_{ii} & \alpha_{ij} \\ \alpha_{ji} & \alpha_{jj} \end{vmatrix} > 0, {}_i \neq {}_j; \quad \begin{vmatrix} \alpha_{ii} & \alpha_{ij} & \alpha_{iK} \\ \alpha_{ji} & \alpha_{jj} & \alpha_{jK} \\ \alpha_{Ki} & \alpha_{jK} & \alpha_{KK} \end{vmatrix} < 0, {}_i \neq {}_j \neq {}_K \neq {}_i; \text{ etc.} \tag{18.0}$$

Even if the approach outlined here is not accepted, these conditions are the direct restrictions imposed upon our demand function as the result of the usual stability or " concavity " conditions. The translation of these conditions into terms of elasticity coefficients of price and income is of course always possible, but is somewhat tedious and

[1] In the integrable case, the form will be already symmetrical,

[2] See M. Bocher, *Introduction to Higher Algebra*, p. 150.

is not given here. It is my feeling that much of the modern work has been rendered unnecessarily lengthy and involved because of a preoccupation with elasticity expressions, which are essentially redundant, since it is always possible to utilise a developed mathematical theory in expressing the conditions on the various partial derivatives directly. However, if this is desired, it is possible to make use of the following relationship derivable directly from the homogeneity assumption by Euler's Theorem:

$$N_{i1}+N_{i12}+ \ldots N_{in}+N_{iI}\equiv 0 \qquad (i=1, \ldots n) \qquad (19.0)$$

where the N's stand for elasticity coefficients.

In the case of two commodities our conditions take the following simple form:

$$\left.\begin{array}{l} N_{11}+N_{12}+N_{1I}=0 \\ N_{21}+N_{22}+N_{2I}=0 \end{array}\right\} \qquad (20.0)$$

and

$$\left.\begin{array}{l} N_{11}+k_1N_{1I}<0 \\ N_{22}+k_2N_{2I}<0 \end{array}\right\} \qquad (20.1)$$

where k represents the proportion of total income spent on the respective commodities.

Concerning definitions of complementarity, I have little to say. It is my personal opinion that the subject has received more attention than would seem justifiable. In other isomorphic equilibrium systems, e.g., the equations of analytic dynamics, or the Gibb's system of thermodynamic equilibrium, it is not felt to be necessary to define similar measures.

Historically, of course, the study of complementarity has been of great importance, since it was in the pursuance of this subject that the inconsistency in the thought of Pareto and the redundancy of the utility concept was revealed. Pedagogically, its introduction may be very desirable.

In concluding this exposition, it may be well to sound a warning. Woe to any who deny any one of the three postulates here! For they are, of course, deducible as theorems from the conventional analysis. They are less restrictive than the usual set-up, and logically equivalent to the reformulation of Hicks and Allen. It is hoped, however, that the orientation given here is more directly based upon those elements which must be taken as *data* by economic science, and is more meaningful in its formulation. Even if this will not be granted, the results given in (18.0) are a useful extension of the restrictions in the older analysis, being directly related to the demand functions.

I have tried here to develop the theory of consumer's behaviour freed from any vestigial traces of the utility concept. In closing I should like to state my personal opinion that nothing said here in the field of consumer's behaviour affects in any way or touches upon at any point the problem of welfare economics, except in the sense of revealing the confusion in the traditional theory of these distinct subjects.